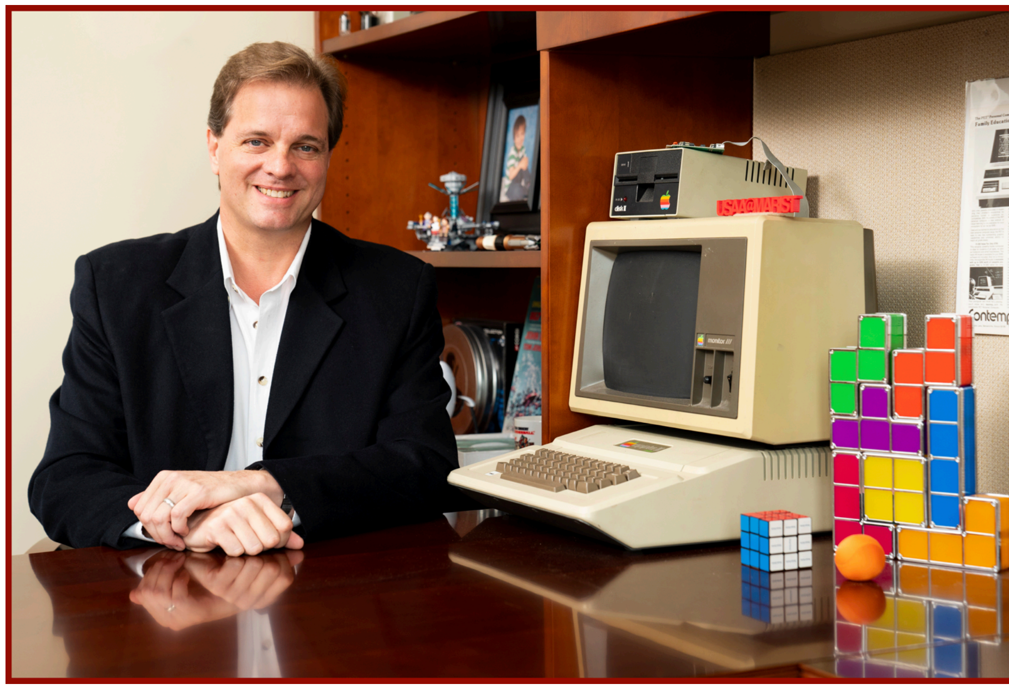
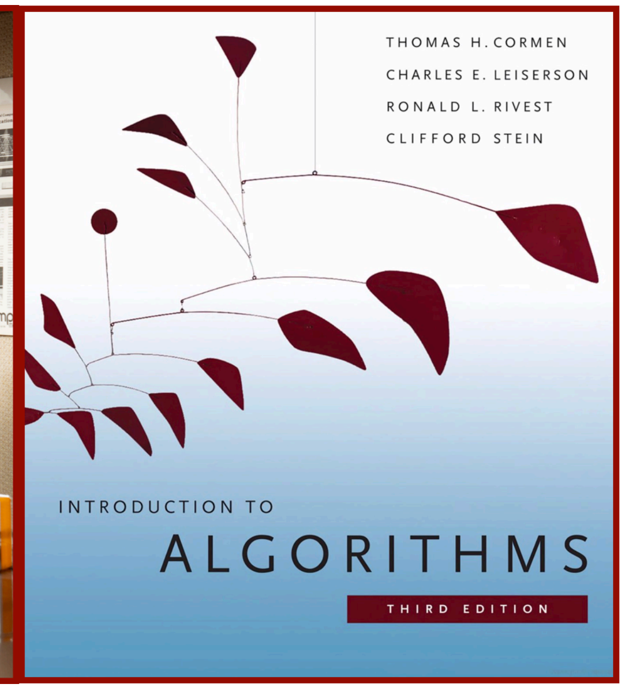
Growth Functions





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Growth Functions

Remember, algorithms are **general recipes** for solving problems **not** specific of any language or platform. We characterize their performance in *time/speed/effort/complexity* with growth functions.

Examples:

$O(n)$	"Order **n**" or "Big-oh of **n**"
$O(n^2)$	"Order **n squared**" or "Big-oh of **n squared**"
$O(\log_2 n)$	"Order **log to the base two of n**" or . . .

We only want the largest (or *dominant*) function of *n* and we ignore constant factors.

$\frac{1}{2} n^2 + 2112$	is $O(n^2)$
$42 n^{1.5} - 8,675,309$	is $O(n^{1.5})$
$11 n \log_2 n + 1$	is $O(n \log_2 n)$
$42 n^{1.5} + \sqrt{n}$	is $O(n^{1.5})$ because $\sqrt{n} = n^{0.5}$ so $n^{1.5}$ dominates

Growth Functions

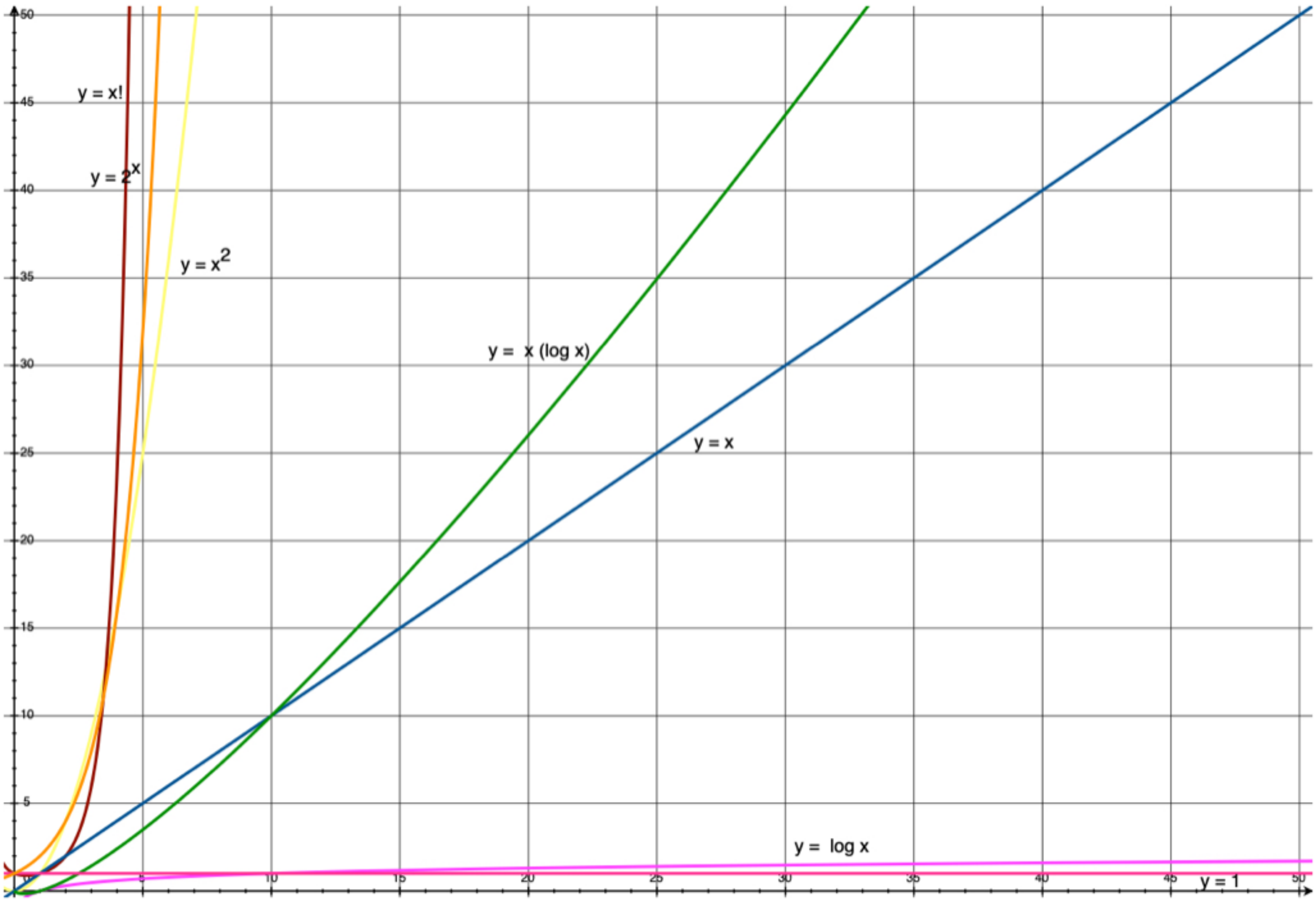
Growth functions let us characterize how the *time/effort/space* required to execute the algorithm grows as the size of the input grows.

Think of this as "complexity".

We're concerned with the measures of effort/complexity needed to correctly solve a problem.

We're also concerned with how those measures change proportionally with the size of the input. I.e., how does the effort scale or grow with the input? What is its "*order of growth*"?

Common Growth Functions



Common Growth Functions

Put in other terms . . .

- Asymptotic Notation: ignore constant factors and low order terms
  - Upper bounds (O), lower bounds (Ω), tight bounds (Θ) $\in, =$, is, order
  - Time estimate below based on one operation per cycle on a 1 GHz single-core machine
  - Particles in universe estimated $< 10^{100}$

input	constant	logarithmic	linear	log-linear	quadratic	polynomial	exponential
n	$\Theta(1)$	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n \log n)$	$\Theta(n^2)$	$\Theta(n^c)$	$2^{\Theta(n^c)}$
1000	1	≈ 10	1000	$\approx 10{,}000$	1,000,000	1000^c	$2^{1000} \approx 10^{301}$
Time	$1\,ns$	$10\,ns$	$1\,\mu s$	$10\,\mu s$	$1\,ms$	$10^{3c-9}\ s$	10^{281} millenia

Millisecond	10^{-3} = 1/1,000
Microsecond	10^{-6} = 1/1,000,000
Nanosecond	10^{-9} = 1/1,000,000,000

A Quick log Refresher

Thanks to Mary Wootters.

Assume log means $\log_2$.

Definition: $\log(n)$ is the number so that $2^{\log(n)} = n$.

In other words, $\log(n)$ is the number of times you need to divide n by 2 to get down to 1.

halve 5 times

$\log_2(32) = 5$ because $32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

$\log_2(64) = 6$ because $64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

halve 6 times

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halve 6 times

$\log_2(128)$	$= 7$	and	$2^7 = 128$
$\log_2(256)$	$= 8$	and	$2^8 = 256$
$\log_2(512)$	$= 9$	and	$2^9 = 512$
$\log_2(1024)$	$= 10$	and	$2^{10} = 1024$

$\log_2$(number of particles in the universe) < 280 so log(n) grows **very** slowly.

Worst Case Analysis

The "running time" (*time/speed/effort/complexity*) of an algorithm is determined by its worst possible input.

In other words, if we say some recipe is an $O(n^2)$ algorithm, that means that the worst possible running time is proportional to n^2 and never worse than that. It could — under lucky circumstances — be better (faster) than $O(n^2)$, but never worse no matter what.

A common example of where we need to apply this thinking is in sorting lists.

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A common example of where we need to apply this thinking is in sorting lists.

Q: Which input to a sort algorithm is worse?

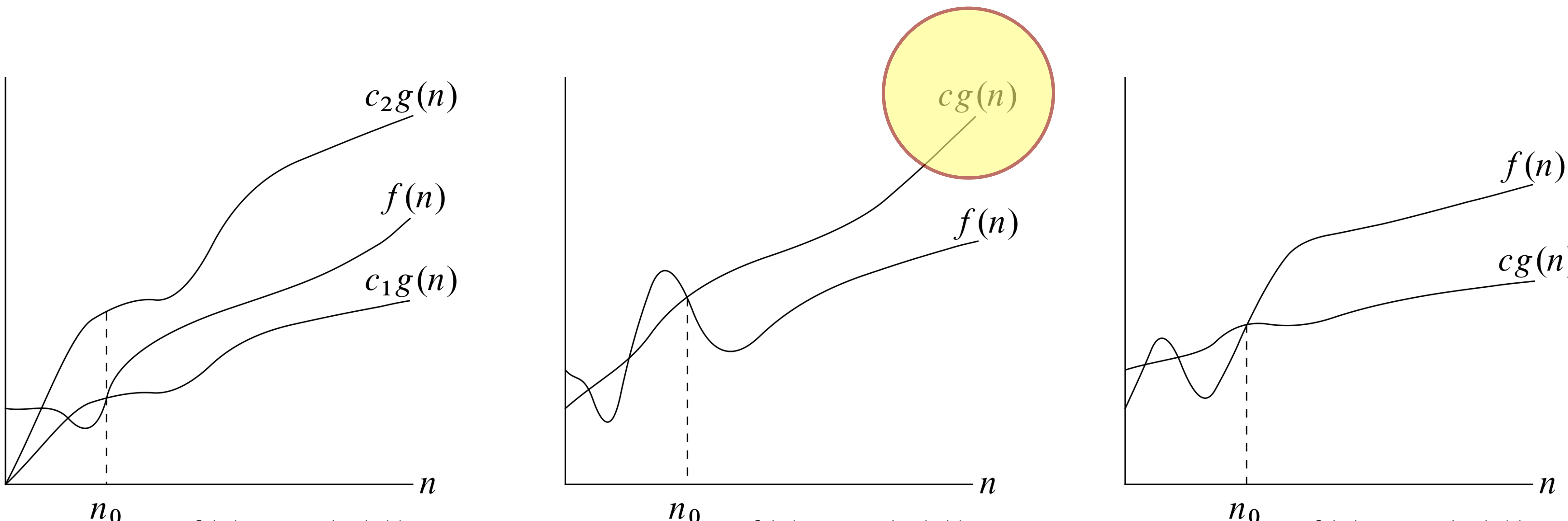
- the elements of the list are "arranged" randomly
- the elements of the list are already sorted in ascending order
- the elements of the list are already sorted in descending order

A: It depends on the specifics of the sorting algorithm.

But when we characterize the sorting algorithm as O(*something*), that must represent the worst-case input.

Asymptotic Analysis

From the CLRS text, section 3.1



$f(n) = \Theta(g(n))$ | $f(n) = O(g(n))$ | $f(n) = \Omega(g(n))$

? | "Big Oh" upper-bound worst case | ?

Asymptotic Analysis

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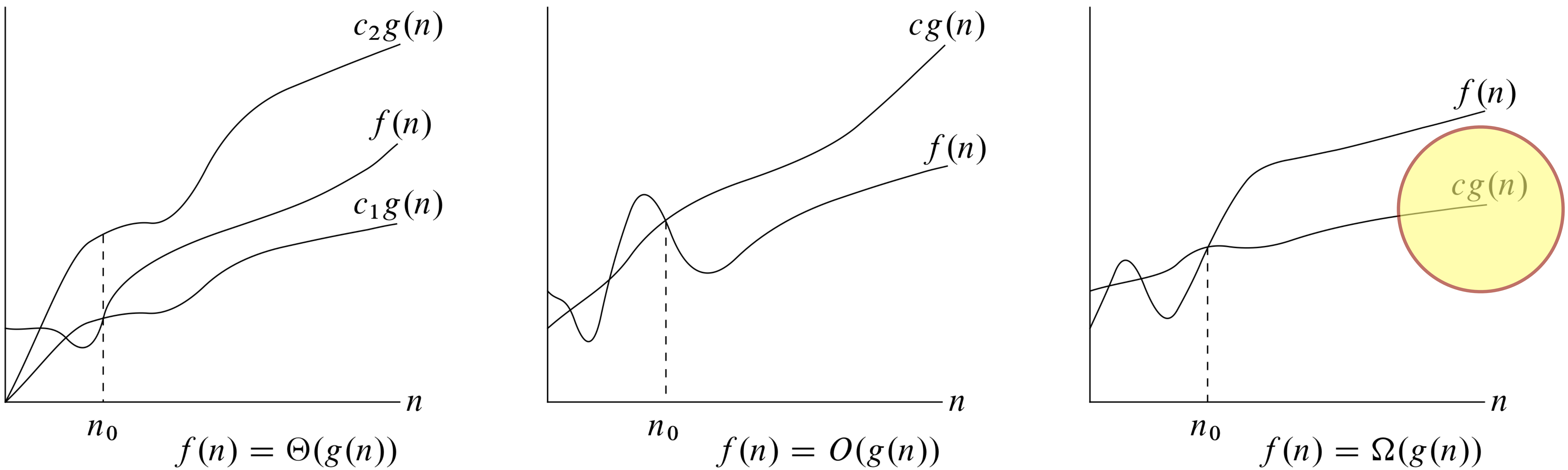

?

"Big Oh"
upper-bound
worst case

"Big Omega"
lower-bound
best case

Asymptotic Analysis

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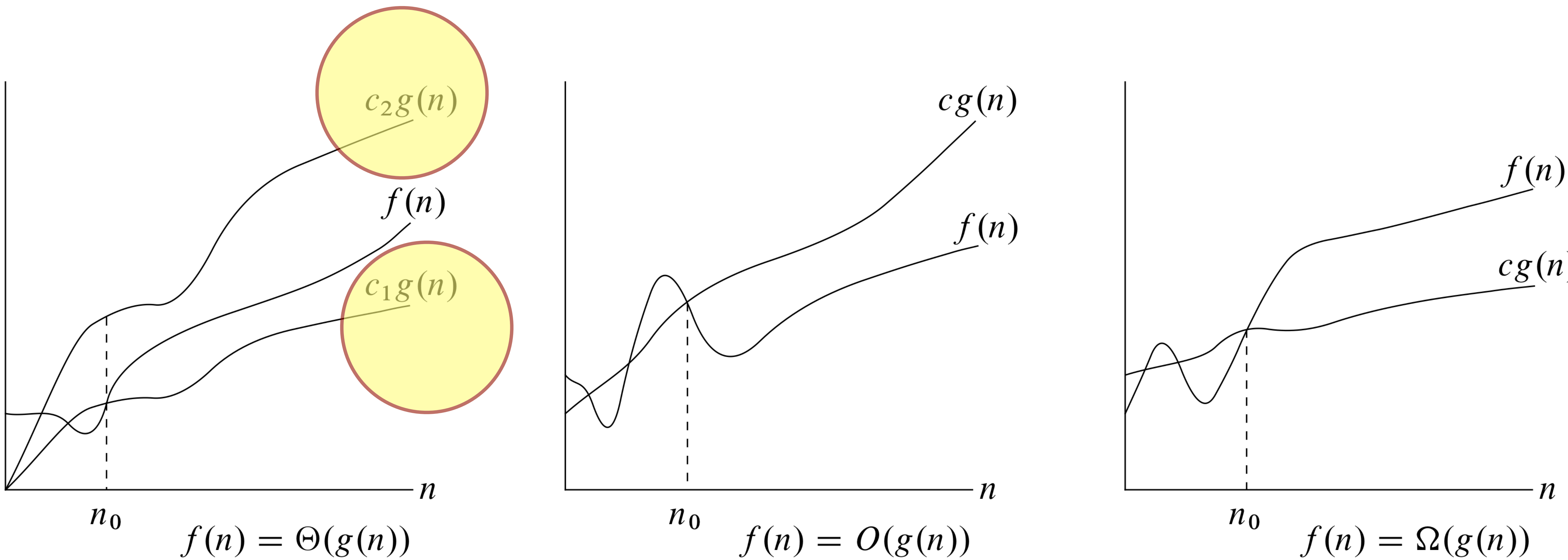

"Big Theta"	"Big Oh"	"Big Omega"
tight-bound	upper-bound	lower-bound
worst and best range	worst case	best case

Asymptotic Analysis :: Big Oh

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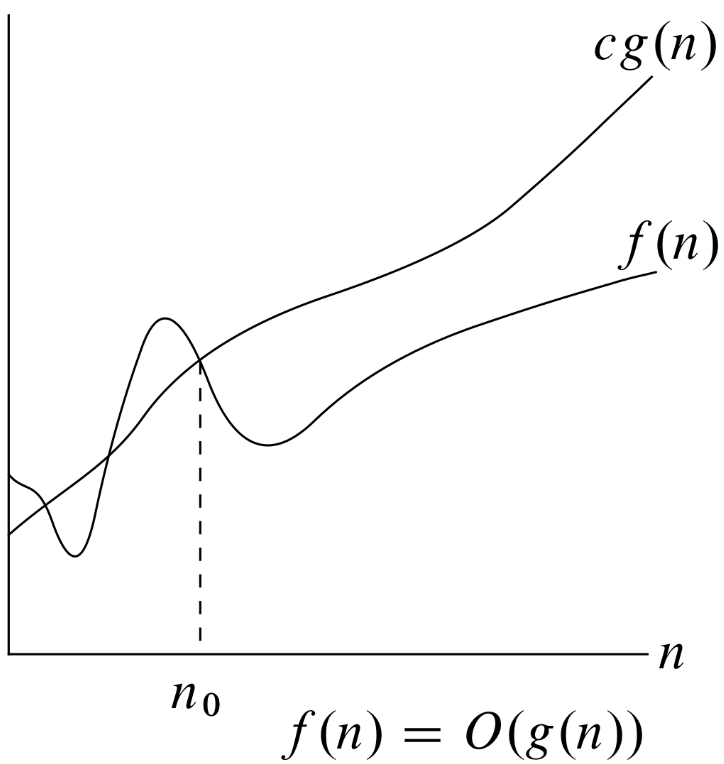

"Big Oh"
upper-bound
worst case

$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \} .$$

Asymptotic Analysis :: Big Oh

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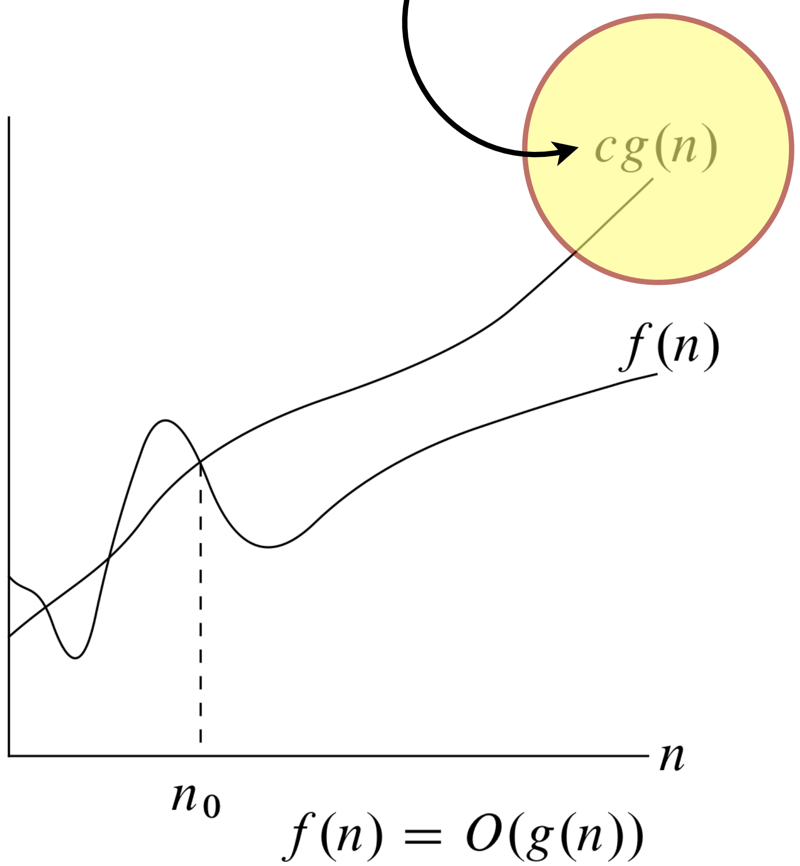

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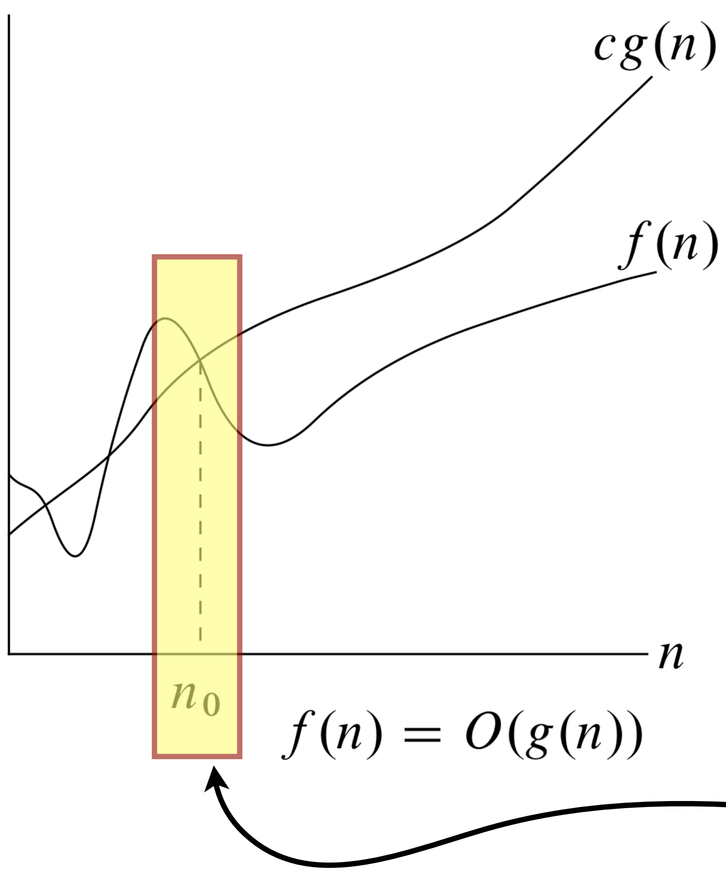

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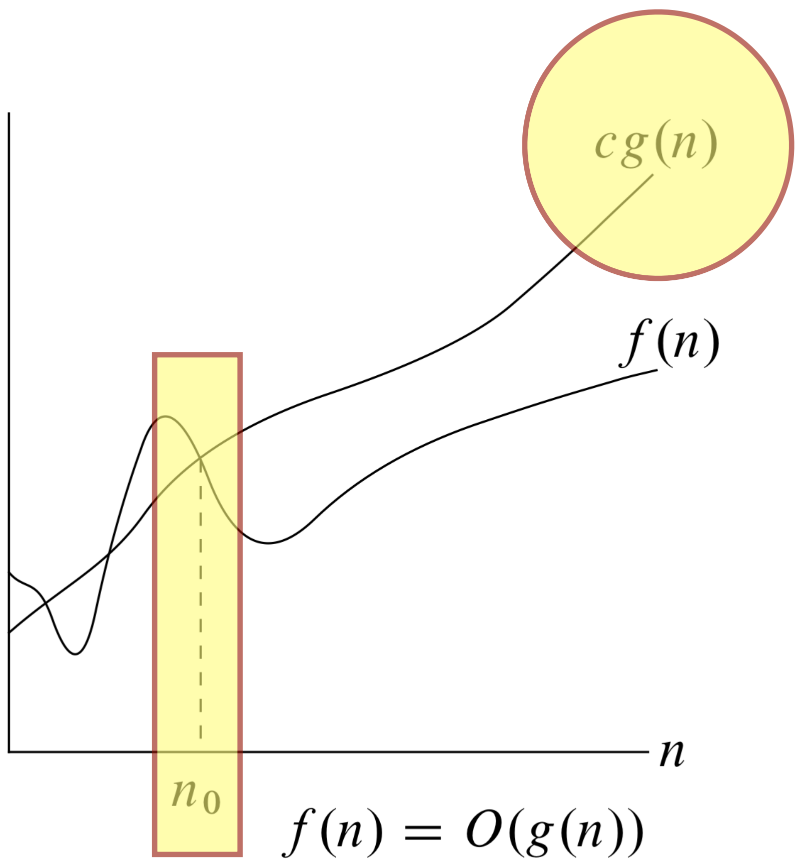

"Big Oh"
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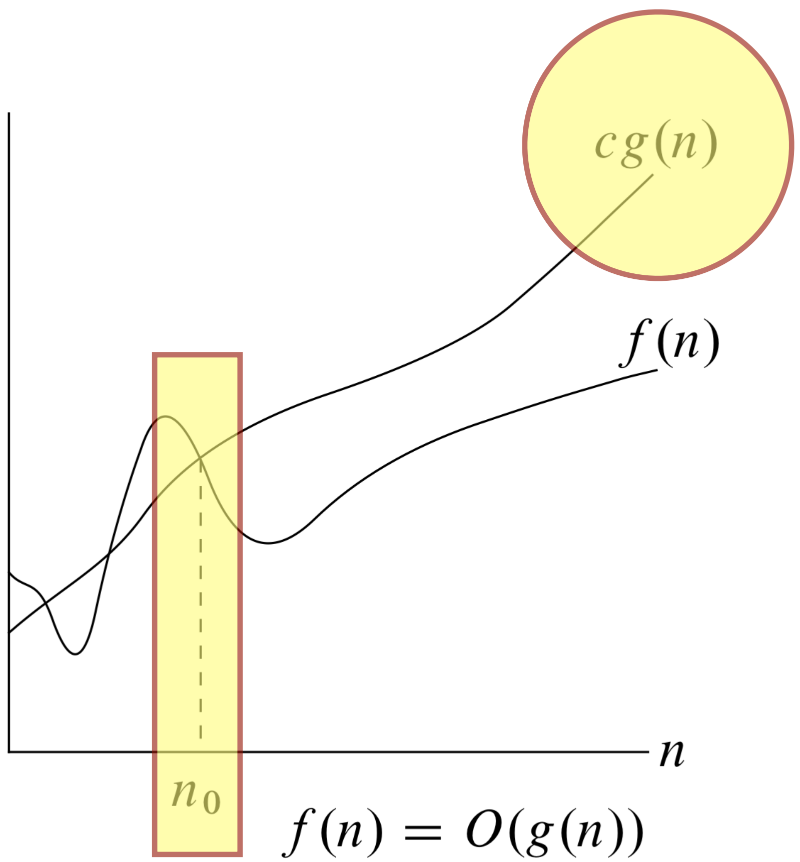

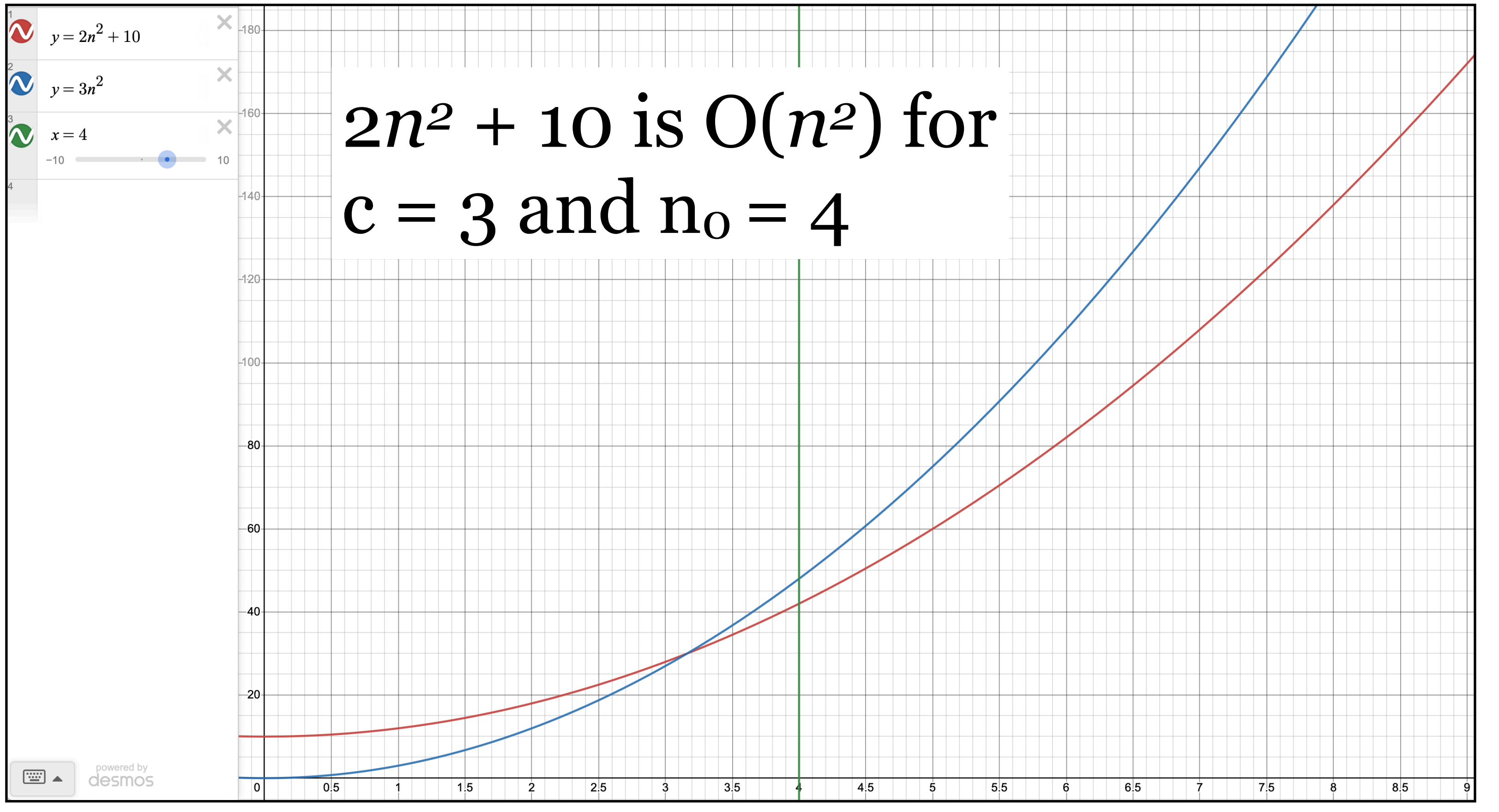

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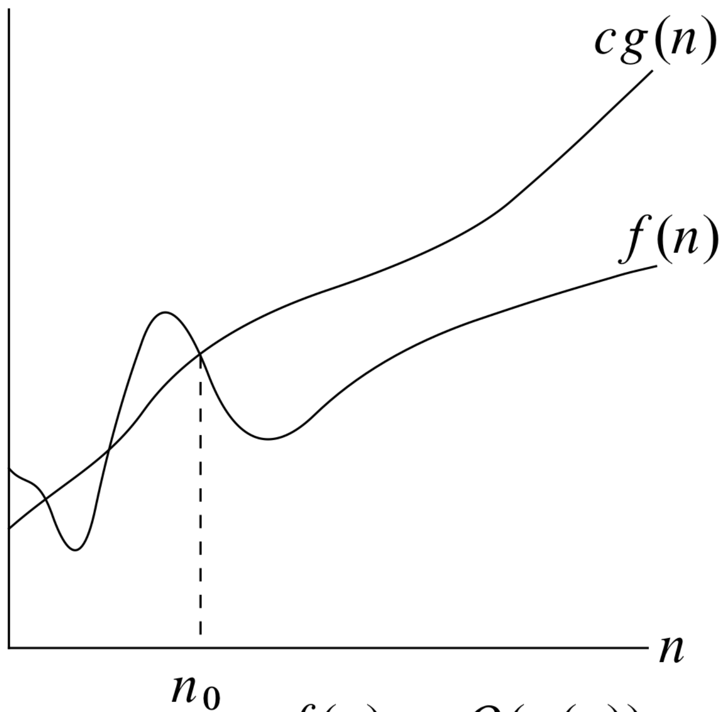

$f(n) = O(g(n))$

“Big Oh”
upper-bound
worst case

$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } \boxed{0 \le f(n) \le cg(n)} \text{ for all } n \ge n_0 \} .$$

Asymptotic Analysis :: Big Omega

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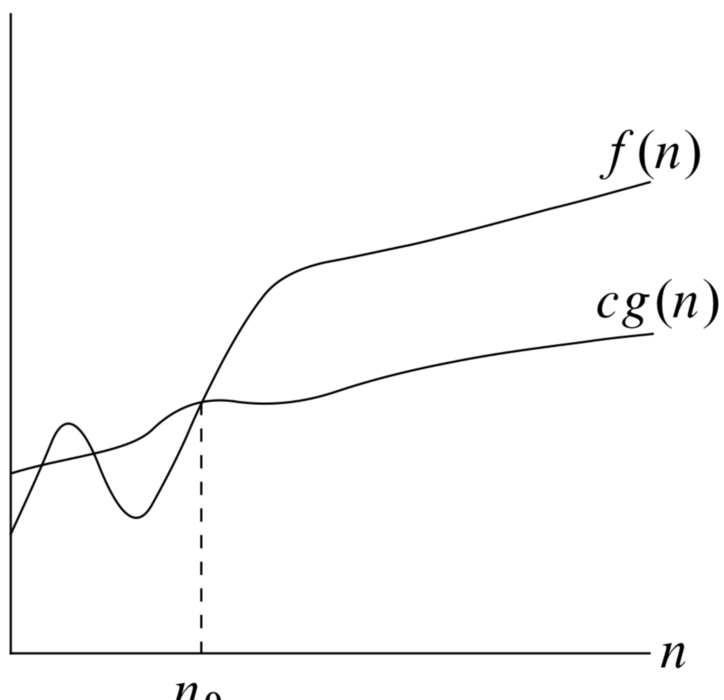

$f(n) = \Omega(g(n))$

"Big Omega"
lower-bound
best case

$$\Omega(g(n)) = \{ f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } \boxed{0 \leq cg(n) \leq f(n)} \text{ for all } n \geq n_0 \} .$$

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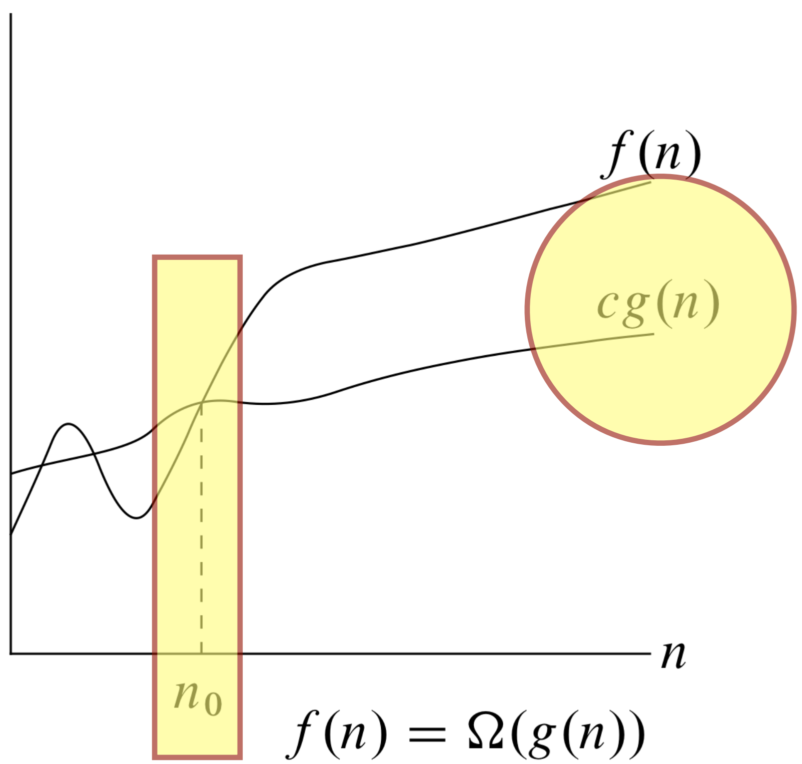

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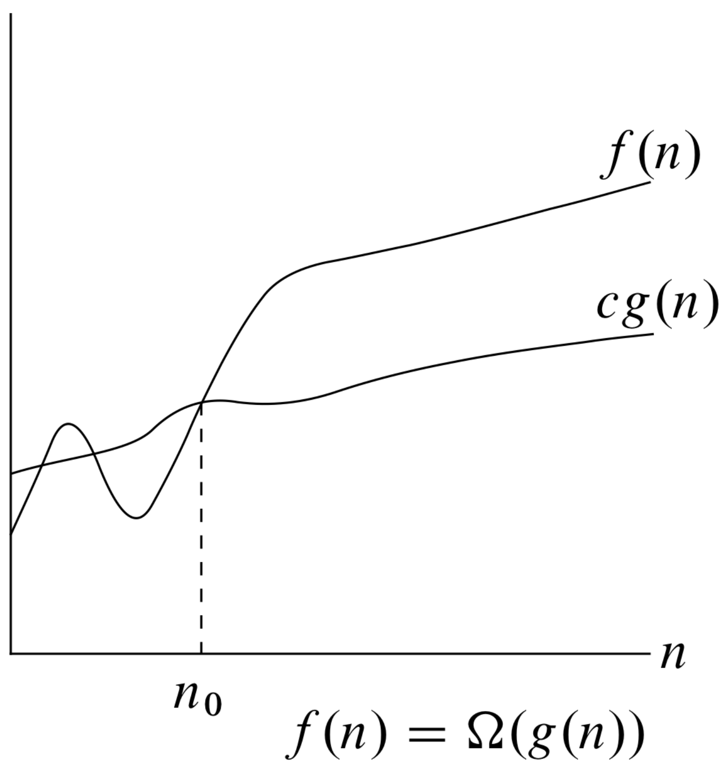

"Big Omega"
lower-bound
best case

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Asymptotic Analysis :: Big Theta

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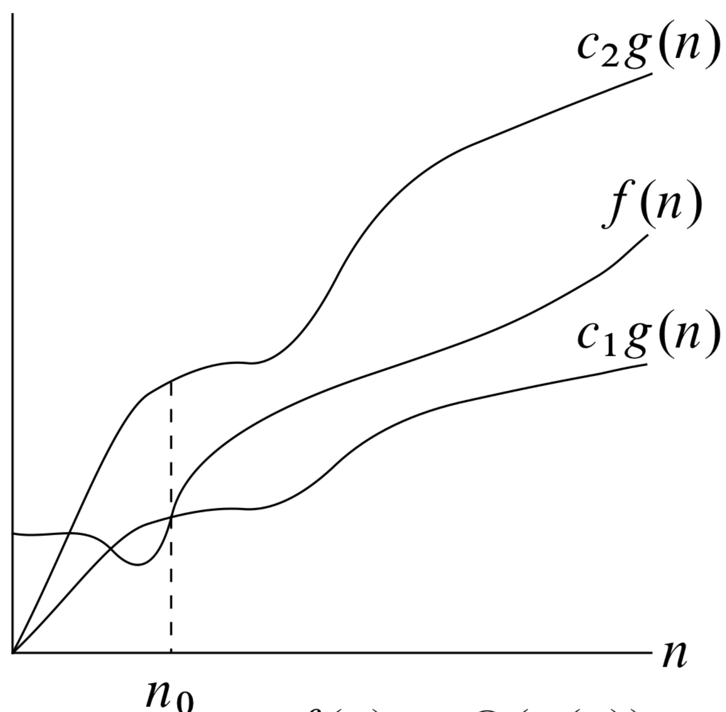

$f(n) = \Theta(g(n))$

"Big Theta"
tight-bound
worst and best range

$$\Theta(g(n)) = \{ f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0 \} .$$

Asymptotic Analysis :: Big Theta

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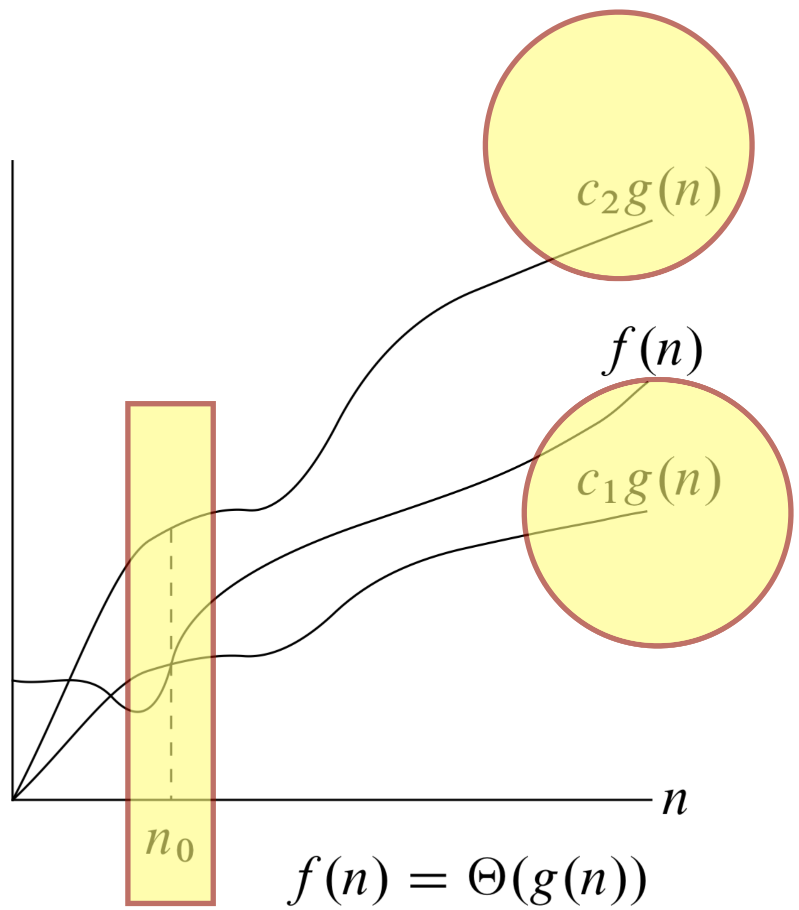

$f(n) = \Theta(g(n))$

"Big Theta"
tight-bound
worst and best range

$\Theta(g(n)) = \{ f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$.

Asymptotic Analysis and Growth Functions

Let's do more examples.