Graph ... as Matrix

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & . & 1 & . & . & 1 & 1 & . \\
2 & 1 & . & 1 & . & 1 & 1 & . \\
3 & . & 1 & . & 1 & . & . & . \\
4 & . & . & 1 & . & 1 & . & . \\
5 & 1 & 1 & . & 1 & . & 1 & 1 \\
6 & 1 & 1 & . & . & 1 & . & 1 \\
7 & . & . & . & . & 1 & 1 & . \\
\end{array}
\]
Graph . . .

as Adjacency List

[1] 2 5 6
[2] 1 3 5 6
[3] 2 4
[4] 3 5
[5] 1 2 4 6 7
[6] 1 2 5 7
[7] 5 6
Independent Sets

An independent set in a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that, for all $u, v$ in $V'$, the edge $\{u, v\}$ is not in $E$. (I.e., no two vertices in $V'$ are adjacent.)
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A maximum independent set is an independent set of the largest possible cardinality.

Can you find one?
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Can you find one? 1 3 7

Can you find two?
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Can you find one? 1 3 7

Can you find two? 1 4 7

Can you find three?
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A **maximum independent set** is an independent set of the largest possible cardinality.

Can you find one? 1 3 7
Can you find two? 1 4 7
Can you find three? 2 4 7
Vertex Cover

A vertex cover of a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $\{u,v\}$ is an edge in $G$, then either $u$ is in $V'$ or $v$ is in $V'$ or both are. (I.e., it’s a set of vertices $V'$ such that each edge of graph $G$ has at least one member of $V'$ as an endpoint.)
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A optimal vertex cover is a vertex cover of minimum size for a given graph.

Can you find one?
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Can you find one? 2 4 5 6

Really?
A vertex cover of a graph \( G = (V, E) \) is a subset \( V' \subseteq V \) such that if \( \{u,v\} \) is an edge in \( G \), then either \( u \) is in \( V' \) or \( v \) is in \( V' \) or both are. (I.e., it’s a set of vertices \( V' \) such that each edge of graph \( G \) has at least one member of \( V' \) as an endpoint.)

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Vertex 2 covers 4 edges.
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  Vertex 4 covers 2 more edges.
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Can you find one?  

2 4 5 6

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Vertex 4 covers 2 more edges.  
Vertex 5 covers 3 more edges.
Vertex Cover

A **vertex cover** of a graph \( G = (V, E) \) is a subset \( V' \subseteq V \) such that if \( \{u,v\} \) is an edge in \( G \), then either \( u \) is in \( V' \) or \( v \) is in \( V' \) or both are. (I.e., it’s a set of vertices \( V' \) such that each edge of graph \( G \) has at least one member of \( V' \) as an endpoint.)

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Can you find one? 2 4 5 6

Really? Let’s test it.
- Vertex 2 covers 4 edges.
- Vertex 4 covers 2 more edges.
- Vertex 5 covers 3 more edges.
- Vertex 6 covers 2 more edges.
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A **optimal vertex cover** is a vertex cover of minimum size for a given graph.

Can you find one? 2 4 5 6

Really? Let’s test it.

- Vertex 2 covers 4 edges.
- Vertex 4 covers 2 more edges.
- Vertex 5 covers 3 more edges.
- Vertex 6 covers 2 more edges.

Yes, really.

Are there others?
A vertex cover of a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $\{u, v\}$ is an edge in $G$, then either $u$ is in $V'$ or $v$ is in $V'$ or both are. (I.e., it’s a set of vertices $V'$ such that each edge of graph $G$ has at least one member of $V'$ as an endpoint.)

A optimal vertex cover is a vertex cover of minimum size for a given graph.

Can you find one? 2 4 5 6
Can you find two? 2 3 5 6
Can you find three?
Vertex Cover

A vertex cover of a graph \( G = (V, E) \) is a subset \( V' \subseteq V \) such that if \( \{u,v\} \) is an edge in \( G \), then either \( u \) is in \( V' \) or \( v \) is in \( V' \) or both are. (I.e., it’s a set of vertices \( V' \) such that each edge of graph \( G \) has at least one member of \( V' \) as an endpoint.)

A optimal vertex cover is a vertex cover of minimum size for a given graph.

Can you find one? 2 4 5 6
Can you find two? 2 3 5 6
Can you find three? 1 3 5 6
Did you notice a relationship between **vertex cover** and **independent set**?
Vertex Cover and Independent Sets

Did you notice a relationship between vertex cover and independent set?

For any graph $G = (V, E)$, if $V' \subseteq V$ is a vertex cover for $G$ then $V - V'$ is an independent set in $G$. 
A **clique** in an undirected graph $G = (V, E)$
... is a subset of the vertex set $V' \subseteq V$, such that for every two vertices in $V'$ there exists an edge connecting them.
... is the subgraph induced by $V'$ as long as it is complete.
A** clique** in an undirected graph $G = (V, E)$

... is a subset of the vertex set $V' \subseteq V$, such that for every two vertices in $V'$ there exists an edge connecting them.

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A** maximal clique** is a clique that cannot be extended by including even one more adjacent vertex.

Can you find a maximal clique?
A **clique** in an undirected graph $G = (V, E)$ … is a subset of the vertex set $V′ \subseteq V$, such that for every two vertices in $V′$ there exists an edge connecting them.

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Can you find a maximal clique?  1 2 5 6
Clique

A **clique** in an undirected graph $G = (V, E)$ ...
... is a subset of the vertex set $V' \subseteq V$, such that for every two vertices in $V'$ there exists an edge connecting them.
...
... is the subgraph induced by $V'$ as long as it is complete.

A **maximal clique** is a clique that cannot be extended by including even one more adjacent vertex.

Maximal clique: 1 2 5 6

These are highly connected vertices. How connected?
The **clustering coefficient** of a vertex in a graph is the probability that the neighbors of that vertex are also connected to each other.

It’s a measure of the local connectivity or “clique-iness” of a (all or part of a) graph. (I.e., the probability that the adjacent vertices of a given vertex are connected.)

\[
\text{clustering coefficient} = \frac{\text{number of actual edges among a vertex's neighbors}}{\text{number of possible edges among a vertex's neighbors}}
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Let’s calculate the clustering coefficient of vertex 2.

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Vertex 2 is adjacent to vertices 1, 3, 5, and 6.
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Let’s calculate the clustering coefficient of vertex 2.

Vertex 2 is adjacent to vertices 1, 3, 5, and 6. Ignore the rest.
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Vertex 2 is adjacent to vertices 1, 3, 5, and 6. Ignore the rest. Ignore vertex 2 itself, since this is about the neighbors.
Clustering Coefficient

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Let’s calculate the clustering coefficient of vertex 2.

Vertex 2 is adjacent to vertices 1, 3, 5, and 6. Ignore the rest. Ignore vertex 2 itself, since this is about the neighbors.

There are 3 actual edges among the remaining vertices 1, 3, 5, and 6.
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There are $n(n-1)/2$ possible edges among the remaining vertices 1, 3, 5, and 6 = 12/2 = 6.
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There are $n(n-1)/2$ possible edges among the remaining vertices 1, 3, 5, and 6 = 12/2 = 6.

ClusteringCoefficient(vertex 2) = 3/6 = 0.5
Clustering Coefficient

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What’s the clustering coefficient of vertex 4?

number of actual edges among a vertex’s neighbors

number of possible edges among a vertex’s neighbors
Clustering Coefficient

The clustering coefficient of a vertex in a graph is the probability that the neighbors of that vertex are also connected to each other.

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What’s the clustering coefficient of vertex 4? It’s 0.

number of actual edges among a vertex’s neighbors
---
number of possible edges among a vertex’s neighbors
Clustering Coefficient

The **clustering coefficient** of a vertex in a graph is the probability that the neighbors of that vertex are also connected to each other.

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What’s the clustering coefficient of vertex 4? It’s 0.

We could increase it — and this make the graph more “clique-ie” — by closing the 3-4-5 triangle (math pun intended).

This **Triadic Closure** is a common graph operation, and what social networks do to suggest people you may know.
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How about the rest?

\[
\frac{\text{number of actual edges among a vertex's neighbors}}{\text{number of possible edges among a vertex's neighbors}}
\]
Clustering Coefficient

The clustering coefficient of a vertex in a graph is the probability that the neighbors of that vertex are also connected to each other.

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How about the rest?

number of actual edges among a vertex’s neighbors
number of possible edges among a vertex’s neighbors
A graph coloring is a way of coloring the vertices of a graph such that no neighbors vertices share the same color.

The graph coloring problem is accomplishing this with a few colors as possible.
Graph Coloring

A **graph coloring** is a way of coloring the vertices of a graph such that no neighbors vertices share the same color.

The graph coloring problem is accomplishing this with a few colors as possible.

Here is a 4-color solution.

Is there a 3-color solution?
A **graph coloring** is a way of coloring the vertices of a graph such that no neighbors vertices share the same color.

The graph coloring problem is accomplishing this with a few colors as possible.

Here is a 4-color solution.

Is there a 3-color solution?

We can use graph coloring to model Sudoku.
There’s more . . .

- Distribution of vertex degrees
- Distribution of clustering coefficients and triadic closure
- Network density
- Size of connected components
- Shortest distance between pairs of vertices
- The centrality or eccentricity of vertices by various measures (PageRank and closeness centrality are of particular interest.)

But it will have to wait.